

SOME FORMULAS FOR NUMBERS OF RESTRICTED WORDS

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ABSTRACT. We define a quantity $c_m(n, k)$ as a generalization of the notion of the composition of the positive integer n into k parts. We proceed to derive some known properties of this quantity. In particular, we relate two partial Bell polynomials, in which the sequence of the variables of one polynomial is the invert transform of the sequence of the variables of the other.

We connect the quantities $c_m(n, k)$ and $c_{m-1}(n, k)$ via Pascal matrices. We then relate $c_m(n, k)$ with the numbers of some restricted words over a finite alphabet. We develop a method which transfers some properties of restricted words over an alphabet of N letters to the restricted words over an alphabet of $N + 1$ letters. Several examples illustrate our findings.

Note that all our results depend solely on the initial arithmetic function f_0 .

1. INTRODUCTION

In this paper, the author continues the investigation of the properties of restricted words, introduced in two previous papers. In the author's paper [4], for a given initial arithmetic function f_0 , we investigated functions f_1, f_2, \dots such that f_m is the m th inverse transform of f_0 . This paper, and Birmajer et al. [1], consider some cases in which f_m counts the number of restricted words over a finite alphabet. In the present paper, we consider the function $c_m(n, k)$ which, in a natural way, generalizes the notion of the composition of n into k parts. Similar functions have recently been considered by several authors, e.g., Eger [3].

Let f_0 be an arithmetic function. For the positive integer m , we let f_m denote the m th invert transform of f_0 . For non-negative integers n, k , ($1 \leq k \leq n$), we define $c_m(n, k)$ recursively in the following way:

$$c_m(0, 0) = 1, \quad c_m(n, 0) = 0, \quad (n > 0),$$

and

$$(1) \quad c_m(n, k) = \sum_{i=1}^{n-k+1} f_{m-1}(i) c_m(n-i, k-1), \quad (1 \leq k \leq n).$$

Note that throughout the paper letters m, n, k will have the meaning as in this definition of $c_m(n, k)$. The definition is a natural generalization of the recurrence for the number of the standard compositions of n into k parts. Firstly, we derive some basic properties of $c_m(n, k)$. As a consequence, we get a relation between the partial Bell polynomial of variables x_1, x_2, \dots and the partial Bell polynomial of variables y_1, y_2, \dots , when the second sequence of variables is the invert transform of the first. Using Birmajer et al. [1, Corollary 10], we derive a formula connecting $c_m(n, k)$ with $c_{m-1}(n, k)$. The formula may simply be written with the use of the

lower triangular Pascal matrices. We then extend this result to obtain a relation between $c_m(n, k)$ and $c_1(n, k)$.

For the particular case when $f_0(1) = 1$, we develop a method which allows us to derive an interpretation of $c_m(n, k)$ in terms of restricted words, when we know the restricted words counted by $c_1(n, k)$. We finish the paper with a number of examples illustrating our results.

2. SOME BASIC PROPERTIES OF $c_m(n, k)$

We start with a few simple facts. For $n = k$, we have $c_m(n, n) = f_{m-1}(1)c_m(n-1, n-1)$. Continuing the same procedure, we obtain

$$(2) \quad c_m(n, n) = f_{m-1}(1)^n.$$

For $k = 1$, we have $c_m(n, 1) = \sum_{i=0}^n f_{m-1}(i)g(n-i, 0)$, that is,

$$(3) \quad c_m(n, 1) = f_{m-1}(n).$$

We next give a simple proof of the fact that $c_m(n, k)$ may be expressed in terms of the partial Bell polynomials $B_{n,k}$.

Proposition 1. *It is true that*

$$(4) \quad c_m(n, k) = \frac{k!}{n!} B_{n,k}(1!f_{m-1}(1), 2!f_{m-1}(2), \dots).$$

Proof. We write Equation (1) in the following way:

$$\frac{n!}{k!} c_m(n, k) = \frac{1}{k} \sum_{i=1}^{n-k+1} f_{m-1}(i) \frac{n!}{(k-1)!} c_m(n-i, k-1).$$

It follows that

$$\frac{n!}{k!} c_m(n, k) = \frac{1}{k} \sum_{i=1}^{n-k+1} i! f_{m-1}(i) \binom{n}{i} \frac{(n-i)!}{(k-1)!} c_m(n-i, k-1).$$

Hence, the quantity $b(n, k) = \frac{n!}{k!} c_m(n, k)$ satisfies the following recurrence:

$$kb(n, k) = \sum_{i=1}^{n-k+1} i! f_{m-1}(i) \binom{n}{i} b(n-i, k-1).$$

Denoting $i! f_{m-1}(i) = x_i$, ($i = 1, 2, \dots$), we obtain the well-known recurrence for the partial Bell polynomials:

$$kB_{n,k}(x_1, x_2, \dots) = \sum_{i=1}^{n-k+1} \binom{n}{i} x_i B_{n-i, k-1}(x_1, x_2, \dots).$$

□

We next prove that $c_m(n, k)$ is a convolution of the sequence $f_{m-1}(1), f_{m-1}(2), \dots$

Proposition 2. *We have*

$$(5) \quad c_m(n, k) = \sum_{i_1+i_2+\dots+i_k=n} f_{m-1}(i_1) f_{m-1}(i_2) \cdots f_{m-1}(i_k),$$

where the sum is taken over positive i_t for $t = 1, 2, \dots, k$.

Proof. Since

$$\begin{aligned} & \sum_{i_1+i_2+\dots+i_k=n} f_{m-1}(i_1)f_{m-1}(i_2)\cdots f_{m-1}(i_k) \\ &= \sum_{i_k=1}^{n-k+1} f_{m-1}(i_k) \sum_{i_1+i_2+\dots+i_{k-1}=n-i_k} f_{m-1}(i_1)f_{m-1}(i_2)\cdots f_{m-1}(i_{k-1}), \end{aligned}$$

the proof easily follows by induction on k . \square

Remark 3. Equation (5) links $c_m(n, k)$ and the weighted integer compositions defined in Eger [3]. Here, the extended binomial coefficients $\binom{k}{n}_f$ are defined to count the number of the so-called weighted compositions of n into k parts, where f is a weighted function. Extended binomial coefficients and $c_m(n, k)$ are connected by

$$c_m(n, k) = \binom{k}{n-k}_{f_{m-1}}.$$

We know that the number of all compositions of n equals the sum of compositions of n into k parts. For the functions $f_m(n)$ and $c_m(n, k)$, we prove the analogous result.

Proposition 4. *The following equation is true:*

$$f_m(n) = \sum_{k=1}^n c_m(n, k).$$

Proof. We use induction on n . For $n = 1$, we have $f_m(1) = c_m(1, 1)$. From Equation (2), it follows that $c_m(1, 1) = f_{m-1}(1)$. Equation $f_m(1) = f_{m-1}(1)$ follows from the author [4, Corollary 2].

Assume that the claim is true for $n - 1$. It follows that

$$\sum_{k=1}^n c_m(n, k) = \sum_{k=1}^n \sum_{i=1}^{n-k+1} f_{m-1}(i) c_m(n-i, k-1).$$

Changing the order of summation on the right-hand side implies

$$\sum_{k=1}^n c_m(n, k) = \sum_{i=1}^n f_{m-1}(i) \sum_{k=1}^{n-i+1} c_m(n-i, k-1).$$

We thus obtain

$$\sum_{k=1}^n c_m(n, k) = f_{m-1}(n) + \sum_{i=1}^{n-1} f_{m-1}(i) \sum_{k=1}^{n-i+1} c_m(n-i, k-1).$$

In the second sum on the right-hand side, the term obtained for $k = 1$ equals zero, which implies that

$$\sum_{k=1}^n c_m(n, k) = f_{m-1}(n) + \sum_{i=1}^{n-1} f_{m-1}(i) \sum_{k=1}^{n-i} c_m(n-i, k-1).$$

Using the induction hypothesis, we obtain

$$\sum_{k=1}^n c_m(n, k) = f_{m-1}(n) + \sum_{i=1}^{n-1} f_{m-1}(i) f_m(n-i),$$

which proves the assertion. \square

Remark 5. The statement of this proposition is obvious for the standard compositions. In our case, it depends on arbitrary initial arithmetic function f_0 . So, we needed a formal proof.

3. A CONNECTION OF $c_m(n, k)$ AND $c_{m-1}(n, k)$

We may view the array $c_m(n, k)$, $(1 \leq k \leq n)$ as a lower triangular matrix $C_m(n)$ of order n , whose (n, k) entry equals $c_m(n, k)$. We let L_n denote the lower triangular Pascal matrix. Hence, the (n, k) entry of L_n is $\binom{n-1}{k-1}$, $(1 \leq k \leq n)$.

First, we prove the following:

Proposition 6. *For each $m > 1$, we have*

$$C_m(n) = C_{m-1}(n) \cdot L_n.$$

Proof. It is easy to see that the statement is equivalent to the following equation:

$$(6) \quad c_m(n, k) = \sum_{i=k}^n \binom{i-1}{k-1} c_{m-1}(n, i).$$

In our terminology, Birmajer et al. [2, Corollary 10] may be written in the form

$$\sum_{j_1+j_2+\dots+j_k=n} f_m(j_1) \cdots f_m(j_k) = \sum_{i=1}^n \binom{i+k-1}{i} c_{m-1}(n, i),$$

where the sum is taken over nonnegative j_1, \dots, j_k . Since at most $k-1$ of j_t may equal 0, Equation (5) yields

$$\sum_{j=0}^{k-1} \binom{k}{j} c_m(n, k-j) = \sum_{i=1}^n \binom{i+k-1}{i} c_{m-1}(n, i).$$

Replacing $k-j$ by t and denoting $\sum_{i=1}^n \binom{i+k-1}{i} c_{m-1}(n, i) = a_k$ implies

$$\sum_{t=1}^k \binom{k}{t} c_m(n, t) = a_k, (k = 1, 2, \dots, n).$$

Denoting $X = (c_m(n, 1), c_m(n, 2), \dots, c_m(n, n))^T$, and $A = (a_1, a_2, \dots, a_n)^T$, this system may be written in the matrix form

$$Q \cdot X = A,$$

where Q is obtained from the Pascal matrix L_{n+1} by omitting the first row and the first column. It follows that $X = Q^{-1} \cdot A$, where $Q^{-1} = \left((-1)^{i+j} \binom{i}{j} \right)_{n \times n}$. For $k = 1, 2, \dots, n$, we obtain

$$(7) \quad c_m(n, k) = \sum_{i=1}^n \left[\sum_{j=1}^n (-1)^{j+k} \binom{k}{j} \binom{i+j-1}{i} \right] c_{m-1}(n, i).$$

Formula (7) holds for each $m > 1$ and for an arbitrary arithmetic function f_0 . In particular, taking $f_0(1) = 1, f_0(i) = 0, (i > 1)$, we obviously have $c_1(n, n) = 1$, and $c_1(n, k) = 0$ for $k < n$. Also $f_1(n) = 1$ for all n . In this case, $c_2(n, k)$ equals

the number of compositions of n into k parts, that is, $c_2(n, k) = \binom{n-1}{k-1}$. Therefore, Equation (7) becomes

$$(8) \quad \binom{n-1}{k-1} = \sum_{j=1}^k (-1)^{j+k} \binom{k}{j} \binom{n+j-1}{n}.$$

The expression in the square brackets in Equation (7) equals $\binom{i-1}{k-1}$, which proves that Equation (10) is true. \square

Remark 7. Note that, as a byproduct, we proved the binomial identity (8).

Remark 8. Replacing $i - k$ by t in Equation (6), we obtain

$$(9) \quad c_m(n, k) = \sum_{t=0}^{n-k} \binom{k+t-1}{t} c_{m-1}(n, k+t).$$

From the equation $C_m(n, k) = C_{m-1}(n, k) \cdot L_n$ follows

$$C_m(n, k) = C_{m-1}(n, k) \cdot L_n = C_{m-2}(n, k) \cdot L_n^2 = \dots = C_1(n) \cdot L_n^{m-1}.$$

We thus obtain

Proposition 9. *The following matrix equation holds*

$$C_m(n) = C_1(n) L_n^{m-1},$$

or, explicitly,

$$(10) \quad c_m(n, k) = \sum_{i=k}^n (m-1)^{i-k} \binom{i-1}{k-1} c_1(n, i), \quad (1 \leq k \leq n).$$

Proof. Since entries of L_n^{m-1} are well known, we easily obtain Equation (10). \square

Now, we derive a formula in which $f_m(n)$ is expressed in terms of $c_1(n, k)$.

Proposition 10. *The following formula holds*

$$(11) \quad f_m(n) = \sum_{i=1}^n m^{i-1} c_1(n, i).$$

Proof. Proposition 4 yields

$$f_m(n) = \sum_{k=1}^n \sum_{i=k}^n (m-1)^{i-k} \binom{i-1}{k-1} c_1(n, i).$$

Changing the order of summation gives

$$f_m(n) = \sum_{i=1}^n \left[\sum_{k=1}^i (m-1)^{i-k} \binom{i-1}{k-1} \right] c_1(n, i).$$

Using the binomial theorem, we obtain Equation (11). \square

Note 11. Equation (11) appears in Birmajer et al. [2], where it is obtained using the properties of the partial Bell polynomials.

As an immediate consequence of Proposition 1 and Equation (10), we obtain the following identity for the partial Bell polynomials.

Identity 12. *If the sequence y_1, y_2, \dots is the invert transform of the sequence x_1, x_2, \dots , then*

$$k!B_{n,k}(y_1, 2! \cdot y_2, 3! \cdot y_3, \dots) = \sum_{i=k}^n \binom{i-1}{k-1} i!B_{n,i}(x_1, 2! \cdot x_2, 3! \cdot x_3, \dots).$$

The following simple result connects the number of some compositions of n into k parts with the number of restricted binary words of length $n-1$ with $k-1$ ones. The restriction, which we denote by \mathcal{R} , is given by the kind of compositions.

Corollary 13. *There is a bijection between compositions of n into k parts and \mathcal{R} -restricted binary words of length $n-1$ with $k-1$ ones.*

Proof. The bijection is given by the following correspondence:

$$(12) \quad 1 \rightarrow 1, 2 \rightarrow 10, 3 \rightarrow 100, \dots$$

In this way, the compositions of n into k parts designate \mathcal{R} -restricted binary word of length n and having k ones, all of which begin with 1. The converse is also true. Omitting the leading 1, we obtain the desired correspondence. \square

We next extend the above result.

Proposition 14. *Take $m > 1$. Assume that $f_0(1) = 1$ and that $f_{m-1}(n)$ counts the \mathcal{R} -restricted words of length $n-1$ over a finite alphabet α . Let x be a letter which is not in α . Then, $c_m(n, k)$ equals the number of \mathcal{R} -restricted words of length $n-1$ over the alphabet $\alpha \cup \{x\}$ in which x appears $k-1$ times.*

Proof. Since $f_0(1) = 1$, it follows from the author [4, Corollary 2] that $f_{m-1}(1) = 1$. From this, one easily obtains that $f_{m-1}(n) > 0$ for all n .

We use induction on k . For $k = 1$, Equation (3) yields $c_m(n, 1) = f_{m-1}(n)$. Since $f_{m-1}(n)$ equals the number of \mathcal{R} -restricted words of length $n-1$ over α , not containing x , we conclude that the statement is true for $k = 1$. Assume that it is true for $k-1$. Consider the first term $f_{m-1}(1)c_m(n-1, k-1) = c_m(n-1, k-1)$ in Equation (1). By the induction hypothesis, $c_m(n-1, k-1)$ equals the number of \mathcal{R} -restricted words of length $n-2$ having $k-2$ letters equal to x . Adding x at the beginning of each such word, we obtain all \mathcal{R} -restricted words of length $n-1$ over $\alpha \cup \{x\}$, having $k-1$ letters equal to x and all of which begin with x .

Consider now the summand $f_{m-1}(i) \cdot c_m(n-i, k-1)$ in Equation (1). By the induction hypothesis, $c_m(n-i, k-1)$ equals the number of \mathcal{R} -restricted words of length $n-i-1$ with $k-2$ letters equal to x . We first insert x at the beginning of each such word. In front of x , we insert an arbitrary word of length $i-1$ over α , which are $f_{m-1}(i)$ in number. We thus obtain all words of length n over $\alpha \cup \{x\}$, such that the first appearance of x is in the i th position. Since the restriction \mathcal{R} does not concern x , it follows that x may be in an arbitrary place in a word. It implies that the right-hand side of Equation (1) counts all the desired words. \square

Remark 15. We stress the fact that the restriction \mathcal{R} concerns only the letters from α . Also, our result essentially depends on the fact that $f_0(1) = 1$.

Now, we illustrate our method by a simple example.

Example 16. Assume that $f_0(i) = 1$ for $i = 1, 2, \dots$. According to Equation (3), we have $f_{m-1}(n) = m^{n-1}$. Hence $f_{m-1}(n)$ equals the number of words of length $n-1$ over the alphabet $\{0, 1, \dots, m-1\}$ with no restriction. Then, $c_1(n, k) = \binom{n-1}{k-1}$.

This means that $C_1(n) = L_n$. It follows that $C_m(n) = L_n^m$. From the well known formula for the terms of L_n^m , we obtain

$$(13) \quad c_m(n, k) = m^{n-k} \binom{n-1}{k-1}.$$

Equation (13) is in accordance with Proposition 14. Namely, according to Proposition 14, $c_m(n, k)$ equals the number of words of length $n-1$ over $\{0, 1, \dots, m\}$ with $k-1$ letters equal to m , and with no restrictions. These $k-1$ letters may be chosen in $\binom{n-1}{k-1}$ ways. The remaining letters may be arbitrary letters from $\{0, 1, \dots, m-1\}$, which are m^{n-k} in number. As a byproduct, using Equation (10), we obtain the following binomial identity:

Identity 17. *For $m > 1$, we have*

$$(14) \quad m^{n-k} \binom{n-1}{k-1} = \sum_{j=0}^{n-k} (m-1)^j \binom{n-1}{k+j-1} \binom{k+j-1}{j}.$$

This simple case is related with to Tchebychev polynomials $U_n(x)$ of the second kind.

Corollary 18. *The expression $[[x^{n-k}](U_{n+k-2}(x))]$ equals the number of words of length $n-1$ over the alphabet $\{0, 1, 2\}$ having $k-1$ twos.*

Proof. It is a well-known fact that $(-1)^k 2^{n-k} \binom{n-1}{k-1}$ is the coefficient of $U_{n+k-2}(x)$ by x^{n-k} . We thus obtain

$$[x^{n-k}](U_{n+k-2}(x)) = (-1)^{n-k} \sum_{j=0}^{n-k} \binom{n-1}{k+j-1} \binom{k+j-1}{j}.$$

This is the case, when in Equation (14), we take $m = 2$. □

4. MORE EXAMPLES

Firstly, we revise the result from the author [5, Corollary 9].

Example 19. We define $f_0(1) = f_0(1) = 1$, and $f_0(n) = 0$ otherwise. According to the author [4, Corollary 33], we know that $f_{m-1}(n)$ equals the number of words of length $n-1$ over the alphabet $\{0, 1, \dots, m-1\}$ having all zeros isolated, which is the restriction \mathcal{R} .

Corollary 20. *The number $c_m(n, k)$ equals the number of words of length $n-1$ over the alphabet $\{0, 1, \dots, m\}$ which have $k-1$ letters equal to m and all zeros are isolated. Also,*

$$c_1(n, k) = \binom{k}{n-k},$$

and

$$c_m(n, k) = \sum_{j=\lceil \frac{n}{2} \rceil - k}^{n-k} (m-1)^j \binom{j+k-1}{k-1} \binom{j+k}{n-j-k}, \left(m > 1, \left\lceil \frac{n}{2} \right\rceil \geq k \right).$$

Proof. The first statement follows from Proposition 14. From the author [5, Corollary 10], it follows that $c_1(n, k+j) = \binom{k+j}{n-k-j}$, $(j = 0, \dots, n-k)$. This implies that $k+j \geq n-k-j$, which yields $2j \geq n-2k$ and $n \geq 2k$. The formula is true according to Equation (10). □

Next, we reexamine the result in the author [4, Corollary 28].

Example 21. We define $f_0(n) = 1$ when n is odd, and $f_0(n) = 0$ otherwise. In this case, $f_m(n)$ equals the number of words of length $n - 1$ over the alphabet $\{0, 1, \dots, m\}$, avoiding runs of zeros of odd lengths. From the author [5, Proposition 24], it follows that

$$c_1(n, k) = \begin{cases} \binom{\frac{n-k}{2} + k - 1}{k-1}, & \text{if } n - k \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

The number $c_1(n, k)$ equals the number of binary words of length $n - 1$ with $k - 1$ ones, avoiding runs of zeros of odd lengths. This follows from bijection (12). We add a short direct proof.

Corollary 22. *The number $c_1(n, k)$ equals the number of binary words of length $n - 1$ with $k - 1$ ones, avoiding runs of zeros of odd lengths.*

Proof. Assume that n and k are of different parities. Since a word of length $n - 1$ with $k - 1$ ones must have $n - k$ zeros, and since $n - k$ is odd, we conclude that such a word must have an odd run of zeros. It follows that $c_1(n, k) = 0$. If n and k are of the same parity, then $n - k$ is even. This means that there are $\frac{n-k}{2}$ pairs of zeros. Of these $\frac{n-k}{2}$ pairs and $k - 1$ ones, we may form $\binom{\frac{n-k}{2} + k - 1}{k-1}$ words of length $n - 1$ having $k - 1$ ones and avoiding runs of zeros of odd lengths. \square

Using induction and Proposition 14, we obtain

Corollary 23. *The number $c_m(n, k)$ equals the number of words of length $n - 1$ over $\{0, 1, \dots, m\}$ with $k - 1$ letters equal to m , avoiding runs of zeros of odd lengths.*

From Equation (10), we obtain an explicit formula for $c_m(n, k)$.

Example 24. We define $f_0(i) = i, (i = 1, 2, \dots)$. The words of length $n - 1$, avoiding 01, are $\{11 \dots 1, 11 \dots 10, \dots, 00 \dots 0\}$. Hence, there are n such words. Applying Proposition 14 several times, we obtain

Corollary 25. *The number $c_m(n, k)$ equals the number of words of length $n - 1$ over $\{0, 1, \dots, m + 1\}$ having $k - 1$ letters equal to $m + 1$ and avoiding 01.*

Corollary 26. *The following formula holds*

$$c_1(n, k) = \binom{n + k - 1}{2k - 1}.$$

Also,

$$c_m(n, k) = \sum_{i=k}^n (m - 1)^{i-k} \binom{i-1}{k-1} \binom{n+i-1}{2i-1}.$$

Proof. For the first part, we use induction on k . For $k = 1$, we have $c_1(n, 1) = \sum_{i=1}^n i c_1(n - i, 0) = n$, since $c_1(n - i, 0) = 1$ when $n = i$, and $c_1(n - i, 0) = 0$ otherwise. This means that the statement is true for $k = 1$. Using induction reduces the problem to the following identity:

$$\binom{n + k - 1}{2k - 1} = \sum_{i=1}^{n-k+1} i \cdot \binom{n + k - i - 2}{2k - 3}, (k > 1).$$

We let X denote the right-hand side of this equation. It follows that

$$X = \sum_{i=1}^{n-k+1} \binom{n+k-i-2}{2k-3} + \sum_{i=2}^{n-k+1} \binom{n+k-i-2}{2k-3} + \cdots + \sum_{i=n-k+1}^{n-k+1} \binom{n+k-i-2}{2k-3}.$$

Applying the horizontal recurrence for the binomial coefficients on each summand yields

$$X = \binom{n+k-2}{2k-2} + \binom{n+k-3}{2k-2} + \binom{2k-2}{2k-2}.$$

Using the horizontal recurrence once more proves the statement.

The formula is true according to Equation (10). \square

Since $\binom{n+k-1}{2k-1}$ obviously equals the number of binary words of length $n+k-1$ with $2k-1$ zeros, we obtain the following Euler type identity:

Identity 27. *The number of binary words of length $n+k-1$ with $2k-1$ zeros equals the number of ternary words of length $n-1$, having $k-1$ letters equal to 2 and avoiding 01.*

The final two examples concern the case $f_0(1) = 0$. Note that in these cases, Proposition 14 can not be used. The first example is an extension of the result in the author [5, Proposition 13].

Example 28. We define $f_0(1) = 0$, and $f_0(n) = 1$ otherwise. It follows from the author [4, Corollary 24] that, for $n > 3$, $f_m(n)$ equals the number of words of length $n-3$ over $\{0, 1, \dots, m\}$, where no two consecutive letters are nonzero. From the author [5, Proposition 13], we obtain $c_1(n, k) = \binom{n-k-1}{k-1}$ for $(1 \leq k \leq \lfloor \frac{n}{2} \rfloor)$, and $c_1(n, k) = 0$ otherwise. Equation (6) implies that $c_m(n, k) = 0$ when $k > \lfloor \frac{n}{2} \rfloor$.

Corollary 29. *For $n > 3$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, the number $c_m(n, k)$ equals the number of words of length $n-3$ over $\{0, 1, \dots, m\}$ with $k-1$ ones, and where all nonzero letters are isolated. An explicit formula for $c_m(n, k)$ is*

$$c_m(n, k) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - k} (m-1)^j \binom{j+k-1}{k-1} \binom{n-k-j-1}{k+j-1}, \left(1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor\right),$$

and $c_m(n, k) = 0$ when $k > \lfloor \frac{n}{2} \rfloor$.

Proof. We know that $c_1(n, k)$ equals the number of compositions of n into k parts, each of which is greater than 1. Using the bijection (12), we conclude that, for $n > 3$, $c_1(n, k)$ equals the number of binary words of length n beginning with 10, ending with 0 and where all ones are isolated. Omitting 10 at the beginning, and 0 at the end of each word, we conclude that $c_1(n, k)$, ($n > 3$) equals the number of binary words of length $n-3$ with $k-1$ ones all of which are isolated. This means that the statement is true for $m = 1$. Assume that it is true for $m-1$. In Equation (9), by the induction hypothesis, $c_{m-1}(n, k+t)$ equals the number of words of length $n-3$ with $k+t-1$ ones, where all nonzero letters are isolated. Replacing t ones with m 's, we obtain the desired words. The number t may be chosen in $\binom{k+t-1}{t}$ ways. Hence, the right-hand side of Equation (9) counts all the desired words.

The formula follows from Equation (10) and the fact that $n-k-j-1 \geq k+j-1$. \square

Example 30. Define f_0 in the following way: $f_0(2) = f_0(3) = 1$, and $f_0(n) = 0$ otherwise. The author in the author [5, Proposition 5] proved that $c_1(n, k) = \binom{k}{n-2k}, (\lceil \frac{n}{3} \rceil \leq k \leq \lfloor \frac{n}{2} \rfloor)$ and $c_1(n, k) = 0$ otherwise.

We know that $c_1(n, k)$ equals the number of compositions of n into k parts equal to either 2 or 3. In other words, for $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, $c_1(n, k)$ equals the number of binary words of length $n - 1$ with $k - 1$ ones, which have the following properties: a word begins with 0 and ends with 0. Also, zero avoids a run of length greater than 2, and all ones are isolated.

Corollary 31. *For $n > 3$, the number $c_m(n, k)$ equals the number of words of length $n - 1$ over the alphabet $\{0, 1, \dots, m\}$ with $k - 1$ ones, which begin and end with 0. Also, 0 avoids a run of length greater than 2 and all nonzero letters are isolated.*

Proof. We showed that the statement is true for $m = 1$. Assume that the statement holds for $m - 1$. Consider the term $\binom{k+t-1}{t} c_{m-1}(n, k+t)$ in Equation (9). The number $c_{m-1}(n, k+t)$ equals the number of the desired words of length $n - 1$ over $\{0, 1, \dots, m - 1\}$ with $k + t - 1$ ones. We replace t of $k + t - 1$ ones with m and then sum over t to obtain $c_m(n, k)$. \square

From Equation (6), it follows that $c_m(n, k) = 0$ if $k > \lfloor \frac{n}{2} \rfloor$. Otherwise, from Equation (10), we obtain

$$c_m(n, k) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (m-1)^j \binom{j+k-1}{k-1} \binom{k+j}{n-2k-2j}, \left(1 \leq k \leq \lfloor \frac{n}{2} \rfloor\right).$$

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